פרק 6- האינטגרל

האינטגרל הלא מסוים:

$$1.\int (3x^2+6x-2)dx = x^3+3x^2-2x+c$$

$$2.\int x(x+2)dx = \frac{x^3}{3}+x^2+c$$

$$3.\int (2+3x^3)^2 dx = \int (4+12x^3+9x^6)dx = 4x+3x^4+\frac{9}{7}x^7+c$$

$$4.\int \frac{dx}{\sqrt[5]{x^2}} = \int x^{-\frac{2}{5}}dx = \frac{5}{3}x^{\frac{3}{5}}+c$$

$$5.\int \frac{dx}{x^2+8} = \frac{1}{\sqrt{8}}arctg(\frac{x}{\sqrt{8}})+c$$

$$6.\int (e^{3x}-e^{2x})^2 dx = \int (e^{6x}-2e^{5x}+e^{4x})dx = \frac{e^{6x}}{6}-2\frac{e^{5x}}{5}+\frac{e^{4x}}{4}+c$$

$$7.\int (1+\sqrt[3]{x^2})^2 dx = \int (1+2x^{\frac{2}{3}}+x^{\frac{4}{3}})dx = x+\frac{6}{5}x^{\frac{5}{3}}+\frac{3}{7}x^{\frac{7}{3}}+c$$

$$8.\int \cos^2 x dx = \int \frac{1}{2}(1+\cos(2x))dx = \frac{x}{2}+\frac{\sin(2x)}{4}+c$$

$$9.\int \frac{dx}{\sqrt{9-x^2}} = \arcsin(\frac{x}{3})+c$$

$$10.\int \tan(10x+5)dx = \frac{-\ln(\cos(10x+5))}{10}+c$$

$$11.\int (\frac{2\sqrt{x}+3x^2}{x})dx = \int (\frac{2}{\sqrt{x}}+3x)dx = 4\sqrt{x}+\frac{3}{2}x^2+c$$

$$12.\int (3x-5)^{12}dx = \frac{(3x-5)^{13}}{3\cdot 13}+c$$

$$13.\int \frac{dx}{\sqrt{x-1}} = 2\sqrt{x-1}+c$$

$$14.\int \sin(3x-1)dx = \frac{-\cos(3x-1)}{3}+c$$

$$15.\int \sqrt{7x+4}dx = \frac{2}{3}\frac{(7x+4)^{\frac{3}{2}}}{7}+c$$

האינטגרל המסוים:

$$1.\int_0^{\frac{\pi}{2}} \cos x dx = \sin x\Big|_0^{\frac{\pi}{2}} = 1$$

$$2.\int_0^1 \frac{dx}{1+x^2} = arctgx\Big|_0^1 = \frac{\pi}{4}$$

$$3.\int_{-1}^2 e^{-x}dx = -e^{-x}\Big|_{-1}^2 = -e^{-2} + e$$

$$4.\int_0^{2\pi} \sin x dx = -\cos x\Big|_0^{2\pi} = 0$$

$$5.\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2 x dx = (\frac{x}{2} - \frac{\sin 2x}{4})\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = (\frac{\pi}{8} - \frac{1}{4}) - (-\frac{\pi}{8} + \frac{1}{4}) = \frac{\pi}{4} - \frac{1}{2}$$

$$6.\int_0^{\frac{\pi}{4}} \tan x dx = -\ln(\cos x)\Big|_0^{\frac{\pi}{4}} = -\ln(\frac{\sqrt{2}}{2})$$

$$7.\int_0^1 e^{2x}dx = \frac{e^{2x}}{2}\Big|_0^1 = \frac{e^2}{2} - \frac{1}{2}$$

$$8.\int_0^2 (2 - x^5)dx = (2x - \frac{x^6}{6})\Big|_0^2 = 4 - \frac{64}{6}$$

$$9.\int_0^2 3\cos(4x)dx = \frac{3\sin(4x)}{4}\Big|_0^2 = \frac{3}{4}\sin(8)$$

$$10.\int_1^2 \frac{e^x - 5x}{3}dx = \frac{e^x - \frac{5x^2}{2}}{3}\Big|_1^2 = \frac{e^2}{3} - \frac{10}{3} - \frac{e}{3} + \frac{5}{6} = \frac{e^2}{3} - \frac{e}{3} - \frac{5}{2}$$

$$11.\int_0^1 \sqrt{5x+4}dx = \frac{2}{3}\frac{(5x+4)^{\frac{3}{2}}}{5}\Big|_0^1 = \frac{2}{15}(9^{\frac{3}{2}} - 4^{\frac{3}{2}}) = \frac{38}{15}$$

$$12.\int_1^8 \frac{1}{\sqrt[3]{x}}dx = \frac{3}{2}x^{\frac{2}{3}}\Big|_1^8 = \frac{3}{2}(8^{\frac{2}{3}} - 1) = 4.5$$

$$13.\int_{-1}^3 (3x^2 - 2x + 1)dx = (x^3 - x^2 + x)\Big|_{-1}^3 = 24$$

$$14.\int_4^5 (\frac{2}{\sqrt{x}} - x)dx = (4\sqrt{x} - \frac{x^2}{2})\Big|_4^5 = 4\sqrt{5} - 12.5 - 8 + 8 = 4\sqrt{5} - 12.5$$

שאלות אתגר:

.15

$$I=\int_x^{x+3} t(5-t)dt=(\frac{5}{2}t^2-\frac{t^3}{3})\Big|_x^{x+3}=\frac{5}{2}((x+3)^2-x^2)-\frac{1}{3}((x+3)^3-x^3)=\frac{5}{2}(6x+9)-3x^2-9x-9\Rightarrow$$

$$I=-3x^2+6x+13.5$$

$$I'=0$$

$$-6x+6=0$$

$$x=1\Rightarrow(1,16.5)$$

$$I''=-6<0\Rightarrow \max$$

.16

$$H(x)=\int_1^x \frac{1}{1+t^2}dt=arctg\,t\Big|_1^x=arcth(x)-\frac{\pi}{4}$$

$$H(1)=0$$

$$H'(1)=\frac{1}{1+x^2}\Big|_1=\frac{1}{2}$$

השימוש באינטגרל בקינמטיקה:

1. (1) א.

$$a(t)=6\frac{m}{s^2}$$

$$x(t)=\int v(t)dt=\int(6t+5)dt=3t^2+5t+c$$

$$x_0=x(t=0)=0\Rightarrow c=0$$

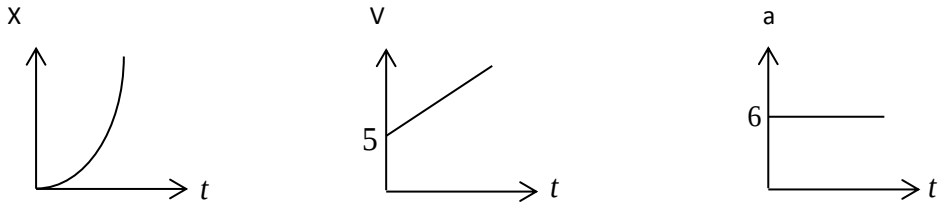
ב.

$$v_0=v(t=0)=5\frac{m}{s}$$

$$a_0=6\frac{m}{s^2}$$

ג. יש להציב $x(t=5),x(t=12),x(t=18.5),a(t=5),a(t=12),a(t=18.5)$

ד.



$$a(t)=40\cos(8t)$$

$$x(t)=\int v(t)dt=\int(5\sin(8t))dt=-\frac{5\cos(8t)}{8}+c$$

(2) א.

ב.

$$x_0=x(t=0)=0\Rightarrow c-\frac{5}{8}=0, c=\frac{5}{8}$$

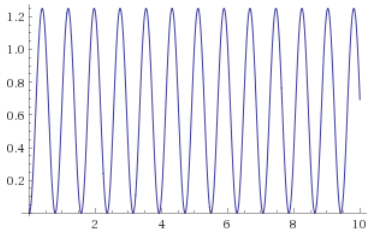
$$x(t)=\frac{5}{8}(1-\cos(8t))$$

$$v_0=v(t=0)=0\frac{m}{s}$$

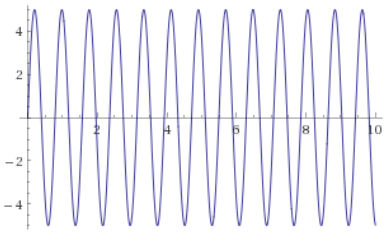
$$a_0=40\frac{m}{s^2}$$

ג. יש להציב $x(t=5), x(t=12), x(t=18.5), a(t=5), a(t=12), a(t=18.5)$

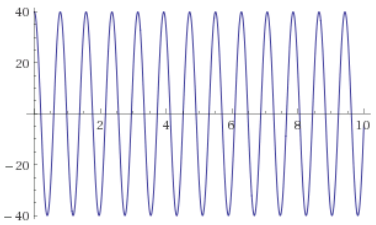
ד.



X,t



V,t



a,t

$$v(t)=\int a(t)dt=\int 2dt=2t+c$$

$$v(t=0)=0\Rightarrow c=0$$

$$v(t)=2t$$

2. (1) א.

$$x(t)=\int v(t)dt=\int(2t)dt=t^2+c$$

$$x(t=0)=0\Rightarrow c=0$$

$$x(t)=t^2$$

$$x_0=0$$

$$v_0=0$$

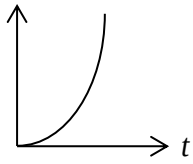
ב.

$$a_0=2\frac{m}{s^2}$$

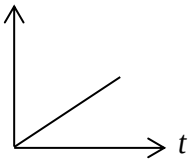
ג. יש להציב $x(t=2), x(t=7), x(t=20), v(t=2), v(t=7), v(t=20)$

ד.

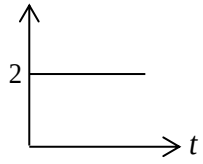
X



V



a



(2) א.

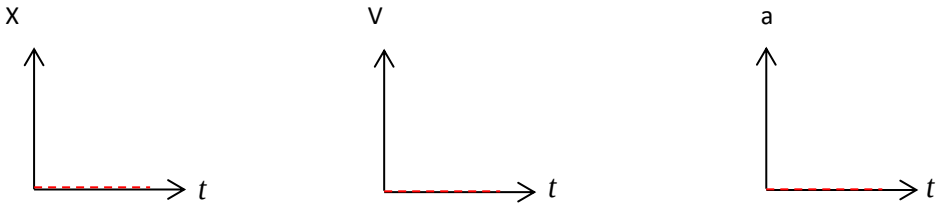
$$v(t)=\int a(t)dt=\int 0dt=c$$
$$v(t=0)=0\Rightarrow c=0$$
$$v(t)=0$$
$$x(t)=\int v(t)dt=\int 0dt=c$$
$$x(t=0)=0\Rightarrow c=0$$
$$x(t)=0$$

ב.

$$x_0=0$$
$$v_0=0$$
$$a_0=0$$

ג. יש להציב $x(t=2),x(t=7),x(t=20),v(t=2),v(t=7),v(t=20)$ אך בכולם נקבל 0.

ד.



(3) א.

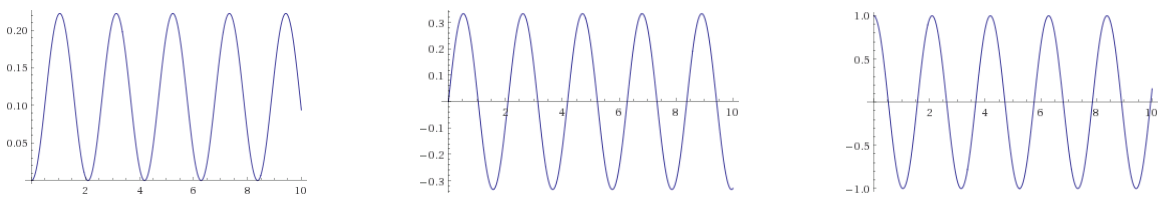
$$v(t)=\int a(t)dt=\int \cos(3t)dt=\frac{\sin(3t)}{3}+c$$
$$v(t=0)=0\Rightarrow c=0$$
$$v(t)=\frac{\sin(3t)}{3}$$
$$x(t)=\int v(t)dt=\int(\frac{\sin(3t)}{3})dt=\frac{-\cos(3t)}{9}+c$$
$$x(t=0)=0\Rightarrow -\frac{1}{9}+c=0,c=\frac{1}{9}$$
$$x(t)=\frac{1}{9}(1-\cos(3t))$$

ב.

$$x_0=x(t=0)=0$$
$$v_0=v(t=0)=0$$
$$a_0=a(t=0)=1\frac{m}{s^2}$$

ג. יש להציב $x(t=2),x(t=7),x(t=20),v(t=2),v(t=7),v(t=20)$

ד.



3. א.

$$x(t)=\int(5t+2)dt=\frac{5}{2}t^2+2t+c$$
$$x(t=0)=0\Rightarrow c=0$$
$$x(t)=\frac{5}{2}t^2+2t$$
$$x(t=5)=72.5m$$

ב.

$$\overline{v}=\frac{x(t=10)-x(t=0)}{10-0}=\frac{270}{10}=27\frac{m}{s}$$

4.

$$v(t)=\int(3+5t)dt=3t+\frac{5}{2}t^2+c$$
$$v(t=0)=10\frac{m}{s}\Rightarrow c=10$$
$$v(t)=3t+\frac{5}{2}t^2+10$$
$$x(t)=\int(3t+\frac{5}{2}t^2+10)dt=\frac{3}{2}t^2+\frac{5}{6}t^3+10t+c$$
$$x(t=0)=15m\Rightarrow c=15$$
$$x(t)=\frac{3}{2}t^2+\frac{5}{6}t^3+10t+15$$

5. נמצא ביטויים כללים למהירותה וההעתקה:

$$v(t)=\int-12tdt=-6t^2+c$$
$$v(t=0)=30\frac{m}{s}\Rightarrow c=30$$
$$v(t)=-6t^2+30$$
$$x(t)=\int(-6t^2+30)dt=-2t^3+30t+c$$
$$x(t=0)=0\Rightarrow c=0$$
$$x(t)=-2t^3+30t$$

כעת נמצא את מרחק הבלימה, בסופו המהירות היא 0:

$$v=0=30-6t^2\Rightarrow t=\sqrt{5}s$$
$$x(\sqrt{5})=30\sqrt{5}-2\cdot5\sqrt{5}=20\sqrt{5}m$$

אינטגרציה על וקטורים:

1.

$$|\vec{v}|=\sqrt{(2t)^2+(5t^2)^2+(3t^3)^2}=\sqrt{4t^2+25t^4+9t^6}$$
$$\hat{v}=\frac{\vec{v}}{|\vec{v}|}=\frac{2t\hat{x}+5t^2\hat{y}+3t^3\hat{z}}{\sqrt{4t^2+25t^4+9t^6}}$$

כדי לראות שהוקטור שקיבלנו הוא באמת וקטור יחידה נציב איזשהו t:

$$\hat{v}(t=2)=\frac{\vec{v}}{|\vec{v}|}=\frac{4\hat{x}+20\hat{y}+24\hat{z}}{\sqrt{992}}$$

$$|\hat{v}|=\frac{\sqrt{16+400+576}}{\sqrt{992}}=1$$

כעת נמצא את ההעתק:

$$\vec{r}(t)=\int\vec{v}(t)dt=\hat{x}\int 2tdt+\hat{y}\int 5t^2dt+\hat{z}\int 3t^3dt=t^2\hat{x}+\frac{5}{3}t^3\hat{y}+\frac{3}{4}t^4\hat{z}+\vec{c}$$

.2

$$\vec{r}(t)=\int\vec{v}(t)dt=\hat{x}\int 6dt+\hat{y}\int 3tdt+\hat{z}\int -10t^2dt=6t\hat{x}+\frac{3}{2}t^2\hat{y}-\frac{10}{3}t^3\hat{z}+\vec{c}$$

$$\vec{r}(t=0)=\vec{c}=5\hat{y}+7\hat{z}$$

$$\vec{r}(t)=6t\hat{x}+(\frac{3}{2}t^2+5)\hat{y}+(7-\frac{10}{3}t^3)\hat{z}$$

.3

$$\vec{v}(t)=\int\vec{a}(t)dt=\hat{x}\int(-2e^{-t})dt=2e^{-t}\hat{x}+\vec{c}$$

$$\vec{v}(t=)=2\hat{x}+\vec{c}=10\hat{x}-7\hat{y}\Rightarrow\vec{c}=8\hat{x}-7\hat{y}$$

$$\vec{v}(t)=(2e^{-t}+8)\hat{x}-7\hat{y}$$

$$\vec{r}(t)=\int\vec{v}(t)dt=\hat{x}\int(2e^{-t}+8)dt+\hat{y}\int-7dt=(-2e^{-t}+8t)\hat{x}-7t\hat{y}+\vec{c}$$

$$\vec{r}(t=1)=(-2e^{-1}+8)\hat{x}-7\hat{y}+\vec{c}=5\hat{y}+3\hat{z}\Rightarrow\vec{c}=(2e^{-1}-8)\hat{x}+12\hat{y}+3\hat{z}$$

$$\vec{r}(t)=(-2e^{-t}+8t+2e^{-1}-8)\hat{x}+(12-7t)\hat{y}+3\hat{z}$$

.4

$$v(t)-v(t=0)=\int_0^t 50\sin(5t+\pi)dt=-10\cos(5t+\pi)\Big|_0^t=-10\cos(5t+\pi)-10$$

$$v(t)=-10(1+\cos(5t+\pi))+2.5=-10\cos(5t+\pi)-7.5$$

$$x(t)-x(t=0)=\int_0^t(-10\cos(5t+\pi)-7.5)dt=(-2\sin(5t+\pi)-7.5t)\Big|_0^t=-2\sin(5t+\pi)-7.5t$$

$$x(t)=-2\sin(5t+\pi)-7.5t+1.23$$